Quantitative Methods in Finance

Step 4: Protect

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Audience and Prerequisites

Audience:

- These notes were prepared for Kenan Flagler Business School's Daytime MBA program
- The setting is a 14 session course
- These notes serve as reference materials to complement our in-class work using computational tools

Prerequisites:

- Some knowledge of probability, statistics are expected
- Knowledge of finance in general, and asset pricing, in particular is expected

Motivating Case Study

- The motivating case study is portfolio allocation
- However, the concepts and tools are widely applicable to a range of settings within Finance
- We group the concepts into the following functional steps within our case study
  1. Explore - Loading and cleaning data, EDA, etc..
  2. Explain - Factor modeling, etc..
  3. Forecast - Time series models, etc...
  4. Protect - Portfolio allocation, risk measurement, etc..

Housekeeping

- Throughout the slide deck you will see “Q”, which indicates a question to you, the reader
- You will also see “A”, which indicates the associated answer
- It is generally most efficient to learn this material through active participation. Whenever you encounter a “Q”, be sure to try and develop your answer before turning the page to the provided “A” answer
- I recommend reading through these slides before engaging in associated coding exercises

Outline

Outline

1. Modern Portfolio Theory
   - Expected Return & Risk of the Portfolio
   - Diversification
   - Markowitz Portfolio Optimization

Reminder on Individual Assets

$$
\begin{aligned}
E[R_i] &= \text{Expected Return on Asset i} \\
V[R_i] \equiv \sigma_i^2 &= E[(R_i - \bar{R}_i)^2] \\
Cov[R_i, R_j] \equiv \sigma_{ij} &= E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)] \\
Corr[R_i, R_j] \equiv \rho_{ij} &= \frac{Cov(R_i, R_j)}{\sigma_i \sigma_j}
\end{aligned}
$$

Portfolio

$$\begin{aligned} E[R_p] &= \sum_{i=1}^{N} w_i E[R_i] \\ V[R_p] \equiv \sigma_p^2 &= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j Cov(R_i, R_j) \\ &= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i<j}^{N} w_i w_j Cov(R_i, R_j) \end{aligned}$$

Portfolio

Example - 2 Assets:

$$
\begin{aligned}
E[R_p] &= w_1 E[R_1] + w_2 E[R_2] \\
V[R_p] &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 Cov(R_1, R_2)
\end{aligned}
$$

Portfolio

Example - 3 Assets:

$$
\begin{aligned}
E[R_p] &= w_1 E[R_1] + w_2 E[R_2] + w_3 E[R_3] \\
V[R_p] &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2 w_1 w_2 Cov(R_1, R_2) \\
&\quad ... + 2 w_1 w_3 Cov(R_1, R_3) + 2 w_2 w_3 Cov(R_2, R_3)
\end{aligned}
$$

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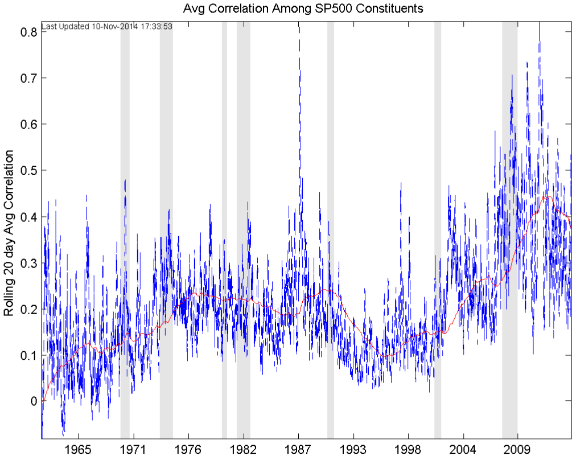
1. Modern Portfolio Theory
   - Expected Return & Risk of the Portfolio
   - Diversification
   - Markowitz Portfolio Optimization

Power of Diversification

- Why do we diversify?
- When is the best time to diversify?



Power of Diversification

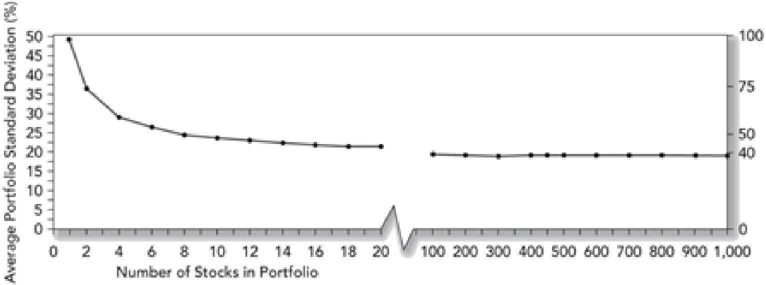


Figure: Statman, "How many stocks in a diversifed portfolio?", J. of Fin'l and Quant. Anal., 1987

Systematic Risk vs Idiosyncratic Risk

Outline

1. Modern Portfolio Theory
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Outline

Weighting Schemes

$$E[R_p] = \sum_{i=1}^{N} w_i E[R_i]$$

$$V[R_p] \equiv = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i<j}^{N} w_i w_j Cov(R_i, R_j)$$

What is the best weighting scheme?

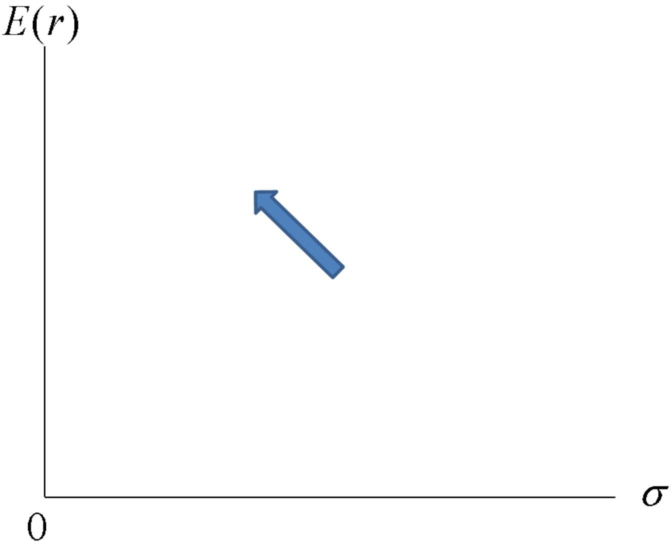
- Equal Weight $w_i = \frac{1}{N}$
- Value Weight $w_i = \frac{P_i \times Shares_i}{\sum_{i=1}^{N} P_i \times Shares_i}$
- Markowitz

Mean Variance Criterion

- Tobin(1958): Only mean and variance are required to describe an investor's preferences for an asset. (Highly debatable)
- Dominance Condition: Security A dominates security B if i) and ii) hold with at least one of i) and ii) being a strict inequality

  i) $E[R_A] \geq E[R_B]$
  ii) $\sigma^2(R_A) \leq \sigma^2(R_B)$
- 2-Moment Utility Function: $U = E[R] - .5\gamma\sigma^2(R)$; $\gamma =$Risk Aversion

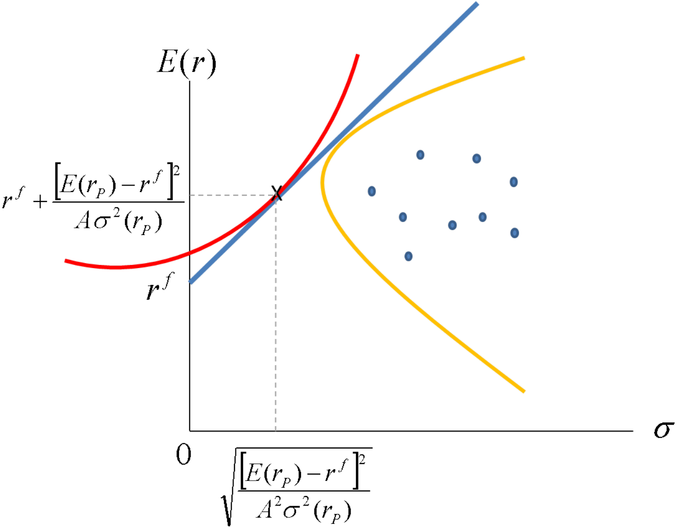
Mean Variance Criterion



We can forecast returns and variance, and place assets inside this space. Then we need to choose the best weighting scheme.

Mean Variance Criterion

There are numerous possible solutions, depending upon the objective function and constraints of the investor.



Outline

Objective Functions

- Minimum Variance subject to target return
- Maximum Return subject to target risk
- Maximum Sharpe Ratio
- Maximum Expected Utility
- Tracking Error
- Factor Exposure Targeting
- etc...

Portfolio Constraints

- Fully Invested
- Short Sales
- Limits on weights
- Limits on factor exposures
- etc....

Min. Variance s.t. Target Return

$$\min_{w} \sigma^2(R_P)$$

$$\sum_{i=1}^{N} w_i = 1$$

$$\sum_{i=1}^{N} w_i E[R_i] = R_p^*$$

Max. Exp. Return s.t. Target Variance

$$\max_{w} E[R_P]$$

$$\sum_{i=1}^{N} w_i = 1$$

$$Var(R_P) = Var(R^*)$$

Max. Sharpe Ratio

$$\max_{w} \frac{E[R_p] - R_f}{\sigma(R_p)}$$

$$\sum_{i=1}^{N} w_i = 1$$

Max. Exp. Utility

$$\max_{w} E[R_p] - .5\gamma\sigma^2(R_p)$$
$$\sum_{i=1}^{N} w_i = 1$$

Min. Tracking Error

Define Tracking Error (TE) $= \sigma(R_p - R_B)$, where R_B is the return on the benchmark portfolio.

$$\min_{w} \sigma(R_P - R_B)$$
$$\sum_{i=1}^{N} w_i = 1$$

Caution: The term Tracking Error is defined in myriad ways. Sometimes $R_p - R_B$ is referred to as tracking error or active return, and $\sigma(R_p - R_B)$ is referred to as tracking risk.

Factor Exposure Targeting

Consider the β for asset i from $R_{i,t} = \alpha + \beta_i SP500_t + u_t$. Now consider an $(N \times 1)$ vector of such betas.

$$\beta = \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_N \end{Bmatrix}$$

What if we want to constrain the beta of the portfolio?

$$0.75 \leq \sum_{i=1}^{N} w_i \beta_i \leq 1.25$$

Factor Exposure Targeting

$$\max_{w} \frac{R_p - R_f}{\sigma(R_p)}$$

$$\sum_{i=1}^{N} w_i = 1$$

$$\underline{\beta} \leq w'\beta \leq \overline{\beta}$$

Factor Exposure Targeting

If we consider multiple factors, we can constrain the exposures across each factor (aka Factor Tilting).

Assume we have N assets and K factors. β_{ik} is asset i's exposure to factor k.

$$B = \left\{ \begin{matrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1K} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2K} \\ \cdots & & & \\ \beta_{N1} & \beta_{N2} & \cdots & \beta_{NK} \end{matrix} \right\}$$

We may add the following constraint

$$\underline{\beta} \leq B'w \leq \overline{\beta}$$

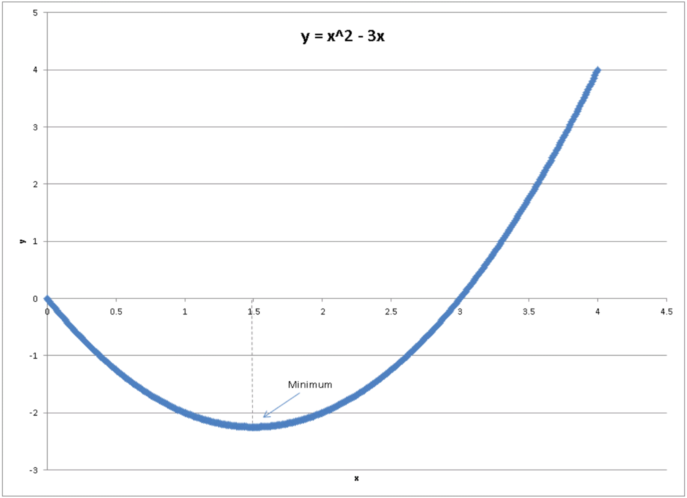
Outline

1. Modern Portfolio Theory
   - Markowitz Portfolio Optimization
     - Foundations
     - Objectives and Constraints
     - Optimization

Optimization

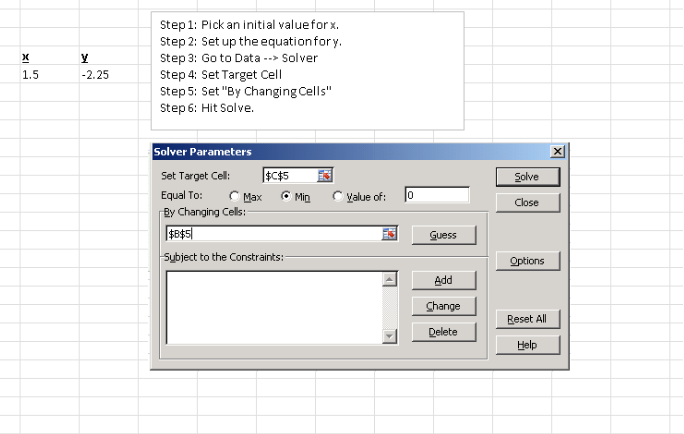
Suppose you are attempting to minimize y by choosing x, where $y = x^2 - 3x$.

You could use calculus, grid search, visualization, etc...to solve.

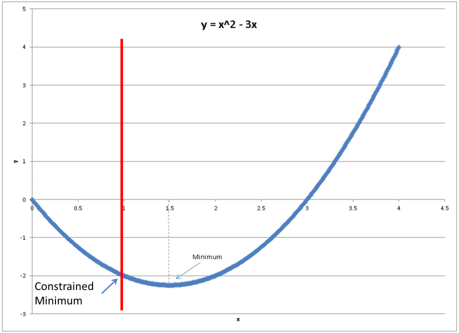


Optimization

Or you could use the Data Solver tool in Excel



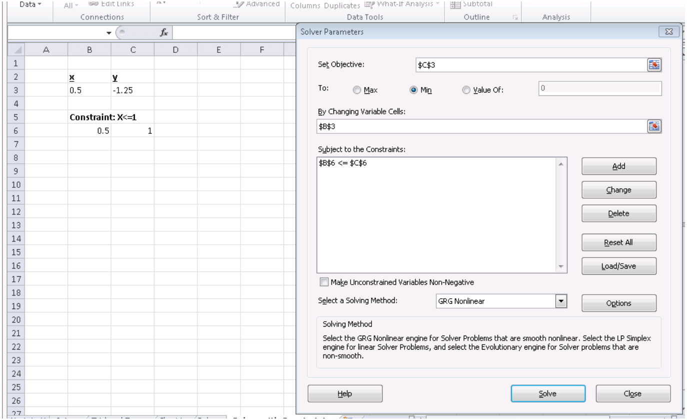
Optimization Constraints



If we constrained the optimization to "Find the value of X that minimizes Y, while keeping X no larger than 1".

We can see easily $X = 1$, generates the minimum value of $Y = -2$.

Optimization Constraints



Matrix Algebra - Portfolio Return

Define R_i as the return on asset i for some given period, and consider N assets. Then portfolio return for that period can be written as $R_p = \sum_{i=1}^{N} R_i$.

For a 3-asset portfolio $N = 3$, define the (3×1) weight vector as $[w_1; w_2; w_3]$, and R as the (3×1) vector of asset returns $[R_1; R_2; R_3]$. We can write in matrix notation

$$R_p = w' \times R = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \times \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

Note: To generate the average portfolio returns, replace the R_i with average returns for asset i (i.e. $\bar{R}_P = \sum_{i=1}^{N} w_i \bar{R}_i$).

Matrix Algebra - Covariance Matrix

Define de-meaned returns as $\tilde{R} = R - \bar{R}$. Suppose $\tilde{R}$ is $(T \times N)$ e.g. (252×3).
Covariance matrix Σ is the $(N \times N)$ matrix defined by $(\tilde{R}'\tilde{R})/(T-1)$.

$$\Sigma = \begin{bmatrix} \tilde{R}_{1,t} & \tilde{R}_{1,t+1} & \dots & \tilde{R}_{1,T} \\ \tilde{R}_{2,t} & \tilde{R}_{2,t+1} & \dots & \tilde{R}_{2,T} \\ \tilde{R}_{3,t} & \tilde{R}_{3,t+1} & \dots & \tilde{R}_{3,T} \end{bmatrix} \times \begin{bmatrix} \tilde{R}_{1,t} & \tilde{R}_{2,t} & \tilde{R}_{3,t} \\ \tilde{R}_{1,t+1} & \tilde{R}_{2,t+1} & \tilde{R}_{3,t+1} \\ \vdots & \vdots & \vdots \\ \tilde{R}_{1,T} & \tilde{R}_{2,T} & \tilde{R}_{3,T} \end{bmatrix} /(T-1)$$

$$= \begin{bmatrix} \sum_{t=1}^{T} \tilde{R}_{1,t}^2 & \sum_{t=1}^{T} \tilde{R}_{1,t}\tilde{R}_{2,t} & \sum_{t=1}^{T} \tilde{R}_{1,t}\tilde{R}_{3,t} \\ \sum_{t=1}^{T} \tilde{R}_{2,t}\tilde{R}_{1,t} & \sum_{t=1}^{T} \tilde{R}_{2,t}^2 & \sum_{t=1}^{T} \tilde{R}_{2,t}\tilde{R}_{3,t} \\ \sum_{t=1}^{T} \tilde{R}_{3,t}\tilde{R}_{1,t} & \sum_{t=1}^{T} \tilde{R}_{3,t}\tilde{R}_{2,t} & \sum_{t=1}^{T} \tilde{R}_{3,t}^2 \end{bmatrix} /(T-1)$$

$$= \begin{bmatrix} Var(R_1) & Cov(R_1, R_2) & Cov(R_1, R_3) \\ Cov(R_2, R_1) & Var(R_2) & Cov(R_2, R_3) \\ Cov(R_3, R_1) & Cov(R_3, R_2) & Var(R_3) \end{bmatrix}$$

Matrix Algebra - Portfolio Variance

For a 3-asset portfolio
$Var(R_p) = w_1^2 Var(R_1) + w_2^2 Var(R_2) + w_3^2 Var(R_3) + 2w_1 w_2 Cov(R_1, R_2) + 2w_1 w_3 Cov(R_1, R_3) + 2w_2 w_3 Cov(R_2, R_3)$.

In matrix notation, define the (3×3) covariance matrix Σ, and define the (3×1) weight vector as $[w_1; w_2; w_3]$.

Then, the variance of the portfolio can be written as

$$Var(R_p) = w'\Sigma w = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \times \begin{bmatrix} Var(R_1) & Cov(R_1, R_2) & Cov(R_1, R_3) \\ Cov(R_2, R_1) & Var(R_2) & Cov(R_2, R_3) \\ Cov(R_3, R_1) & Cov(R_3, R_2) & Var(R_3) \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Outline

1. Modern Portfolio Theory
2. Risk

Risk

- Risk is more than variance (standard deviation)

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- Broadly speaking, risk is the probability that the outcome will be different than expected
- There are myriad measures of risk for specific use cases (e.g. max drawdown, sem-variance, etc..).
- We are going to focus on a common set of measures of tail risk.

VaR & CVaR

- Value at Risk (VaR) is a quantile of the return distribution (Note: often cast as the "loss" distribution").

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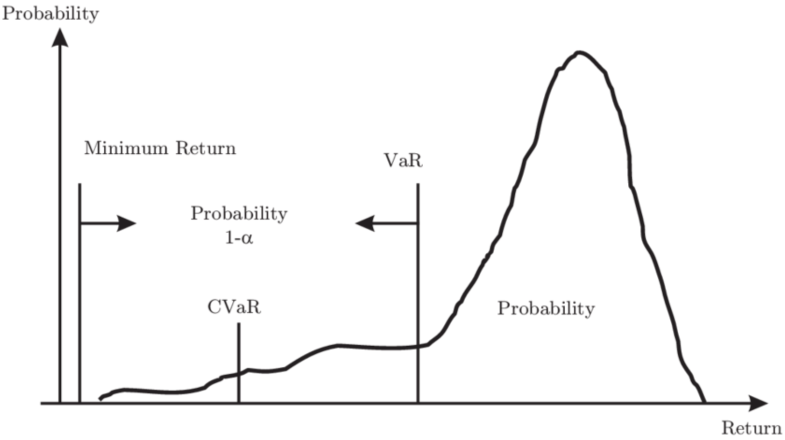
  - Suppose the 95% VaR is -3%. Then we can say that only 5% of the returns are worse than -3%.
  - In this way, the VaR defines the left tail of the return distribution.
- The Conditional Value at Risk (aka Expected Shortfall) tell us the expected tail loss.
  - CVaR is the expected return in the left tail. i.e. if $CVaR = -4.5\%$ then IF the portfolio experiences a extreme loss, we would expect that loss to be -4.5%.

VaR & CVaR



Probability
Minimum Return
VaR
Probability
1-α
CVaR
Probability
Return